Published by Institute of Physics Publishing for SISSA/ISAS
Received: *December 24, 2002*
Accepted: *February 3, 2003*

De Sitter solutions in $N = 4$ matter coupled supergravity

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Abstract: We investigate the scalar potential of gauged $N = 4$ supergravity with matter. The extremum in the SU(1,1)/U(1) scalars is obtained for an arbitrary number of matter multiplets. The constraints on the matter scalars are solved in terms of an explicit parametrisation of an SO$(6, 6+n)$ element. For the case of six matter multiplets we discuss both compact and noncompact gauge groups. In an example involving noncompact groups and four scalars we find a potential with an absolute minimum and a positive cosmological constant.

Keywords: Extended Supersymmetry, Supergravity Models, Cosmology of Theories beyond the SM.

Contents

1. Introduction

The evidence for a positive cosmological constant has led to a renewed interest in gauged supergravity theories. The presence of a scalar potential in these theories opens the possibility of obtaining, at the extremum of the potential, V_0, a nonzero cosmological constant. In the past the interest in this field was concentrated on zero or negative values of V_0, in view of preserving some of the supersymmetries for phenomenological applications.[1] For positive values of V_0, or de Sitter solutions, supersymmetry is necessarily completely broken.

In this paper we will investigate properties of gauged $N = 4$ supergravity in four dimensions with positive V_0. Gauged supergravity itself contains two physical scalars, which take values in the coset $\mathrm{SU}(1,1)/\mathrm{U}(1)$. The potential due to these scalars has been investigated in great detail in the past [3]–[8]. Here we want to extend this work to the case where an arbitrary number of matter multiplets is included.

There are a number of obstructions to the existence of solutions in four-dimensional supergravities with a positive cosmological constant [9, 10, 11]. These obstructions show up in particular in theories obtained by dimensional reduction from eleven or ten dimensions. However, in four dimensions much more is possible. In $N = 4$ supergravity one can for instance add additional matter beyond what would be expected from string theory, one can gauge some of the global symmetries of the supergravity theory, and one can introduce additional parameters [7], called $\mathrm{SU}(1,1)$ angles, which give the matter multiplets different $\mathrm{SU}(1,1)$ orientations. Although gauged supergravities may be related to Scherk-Schwarz reductions of higher dimensional theories, it is then still not clear how to introduce the

[1] For recent work in this direction see, e.g., [1, 2].

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SU(1,1) angles. For $N = 2$ theories a recent investigation [12] revealed that it is possible to obtain stable de Sitter vacua. The analogue of the $N = 4$ SU(1,1) angles played a crucial role in [12].

An interesting investigation of scalar potentials in extended supergravity theories was performed in [13]. These authors also consider $N = 4$, but without additional matter. They remark that in all cases considered there is a simple relation between the value of the potential at its extremum and the masses of the scalar excitations at the extremum:

$$V_0 \simeq \left.\frac{\partial^2 V}{\partial x^2}\right|_0 . \tag{1.1}$$

This relation makes these examples less suitable for application in the context of inflationary models, although in some scenarios they do not seem to be excluded (see [13] for a discussion on this matter). In section 2 we will make this relationship more precise for the SU(1,1) scalars.

In this paper we further develop the formalism of gauged $N = 4$ supergravity with matter, and give examples with positive extrema of the potential. In the remainder of this Introduction we will present some basics of gauged $N = 4$ supergravity. The analysis of the dependence on the SU(1,1) scalars is done in section 2. The matter scalars are considered in section 3. We solve the constraint for these scalars, and discuss properties of the potential for general matter fields. In sections 4 we work out a number of explicit examples for gauge groups $\mathrm{SO}(3)^4$, $\mathrm{SO}(3)^2 \times \mathrm{SO}(2,1)^2$, and $\mathrm{SO}(2,1)^4$. In section 5 we will discuss some additional issues and further work.

We consider gauged $N = 4$ supergravity coupled to n vector multiplets. The bosonic part of the lagrangian density reads [7]: [2]

$$\begin{aligned} e^{-1}\mathcal{L} = &-\frac{1}{2}R + \frac{1}{2}\left(\partial_\mu \phi^\alpha \partial^\mu \phi_\alpha + \phi^\alpha \partial_\mu \phi_\alpha \phi^\beta \partial^\mu \phi_\beta\right) - \frac{1}{2}\eta_{RS}\partial_\mu Z_a{}^R \partial^\mu Z_a{}^S - \\ &-\frac{1}{8}\eta_{RS}\eta_{TU} Z_a{}^R \overleftrightarrow{\partial_\mu} Z_b{}^S \, Z_a{}^T \overleftrightarrow{\partial^\mu} Z_b{}^U - V(\phi, Z) + \\ &+\eta_{RS}\Bigg(-\frac{1}{4}F^{+\,R}_{\mu\nu} F^{\mu\nu\,+\,S} \frac{1}{\Phi_{(R)}}\left(\phi^1_{(R)} - \phi^2_{(R)}\right) - \\ &\qquad - \frac{1}{\Phi_{(R)}} F^{+\,R}_{\mu\nu} K^{\mu\nu\,+\,S} - \frac{1}{2}\frac{\Phi^*_{(R)}}{\Phi_{(R)}} K^{+\,R}_{\mu\nu} K^{\mu\nu\,+\,S} + \text{h.c.}\Bigg) . \end{aligned} \tag{1.2}$$

The scalars ϕ_α ($\phi^1 = (\phi_1)^*, \phi^2 = -(\phi_2)^*$) transform under global SU(1,1) and local U(1), the $Z_a{}^R$ transform under local $\mathrm{SO}(6) \times \mathrm{SO}(n)$, and under global $\mathrm{SO}(6,n)$. The scalars satisfy the constraints

$$\phi^\alpha \phi_\alpha = 1 , \tag{1.3}$$

$$\eta_{RS} Z_a{}^R Z_b{}^S = -\delta_{ab} . \tag{1.4}$$

[2] The indices $\alpha, \beta, \ldots$ take on values 1 and 2, indices $R, S \ldots$ the values $1, \ldots, 6+n$, and the indices $a, b, \ldots$ the values $1, \ldots, 6$. The metric η_{RS} can be chosen as $\mathrm{diag}(-1,-1,-1,-1,-1,-1,+1,\ldots,+1)$, with n positive entries. In comparison to [7] we have replaced the complex scalars $\phi_{ij}{}^R$ by real scalars $Z_a{}^R$: $\phi_{ij}{}^R = \frac{1}{2} Z_a{}^R (G^a)_{ij}$, where the G^a are six matrices which ensure that $Z_a{}^R$ transforms as a vector under SO(6). This redefinition is given (in a slightly different normalisation) in [6].

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Due to these constraints and the local symmetry the scalars are restricted to cosets SU(1,1)/U(1) (two physical scalars) and SO(6,n)/SO(6) × SO(n) (6n physical scalars).

There is a certain freedom in coupling the vector multiplets: for each multiplet we can introduce an SU(1,1) element, of which only a single angle α turns out to be important. These angles α_R appear in the kinetic terms of the vectors in the form

$$\phi^1_{(R)} = e^{i\alpha_R}\phi^1\,, \qquad \phi^2_{(R)} = e^{-i\alpha_R}\phi^2\,, \qquad \Phi_{(R)} = e^{i\alpha_R}\phi^1 + e^{-i\alpha_R}\phi^2\,. \tag{1.5}$$

The gauge group has to be a subgroup of SO(6,n). For a semi-simple gauge group the α_R have to be the same for all R belonging to the same factor of the gauge group. The gauging breaks the global SO(6,n) symmetry of the ungauged theory.

In (1.2) we have made explicit the dependent gauge fields of the local U(1) and SO(6) symmetries. The kinetic terms for the vectors still contain the auxiliary field $T_{\mu\nu}{}^{ij}$, which must be eliminated by solving its equation of motion. Here it is again useful to go to a real basis and to define $T_{\mu\nu}{}^a = \frac{1}{2}T_{\mu\nu}{}^{ij}G^a{}_{ij}$. In this form we have $K_{\mu\nu}{}^R = T_{\mu\nu}{}^a Z_a{}^R$. The equation of motion for T is

$$T_{\mu\nu}{}^a Q_{ab} + \eta_{RS}\frac{1}{\Phi_{(R)}}F^{+\,R}_{\mu\nu}Z_b{}^S = 0\,, \tag{1.6}$$

with

$$Q_{ab} = \eta_{RS}\frac{\Phi^*_{(R)}}{\Phi_{(R)}}Z_a{}^R Z_b{}^S\,. \tag{1.7}$$

The vector kinetic terms can then be written as:

$$\begin{aligned}\mathcal{L}_{\rm vector} = & -\frac{1}{4}\eta_{RS}F^{+R}_{\mu\nu}F^{\mu\nu\,+S}\frac{1}{\Phi_{(R)}}(\phi^1_{(R)} - \phi^2_{(R)}) + \\ & +\frac{1}{2}\eta_{RS}\eta_{TU}\frac{1}{\Phi_{(R)}}F^{\mu\nu\,+R}Z_a{}^S(Q^{-1})_{ab}Z_b{}^T\frac{1}{\Phi_{(T)}}F^{+U}_{\mu\nu} + \text{h.c.}\,.\end{aligned} \tag{1.8}$$

This simplifies in two special cases: firstly, if all SU(1,1) angles are equal ($\Phi_{(R)} = \Phi$) we have $Q_{ab} = -\delta_{ab}\Phi^*/\Phi$, secondly, with six vector multiplets (1.4) can be solved by $Z_a{}^R = \delta_a{}^R$, and $Q_{ab} = -\delta_{ab}\Phi^*_{(a)}/\Phi_{(a)}$.

The scalar potential reads

$$V(\phi, Z) = \left(\frac{1}{4}Z^{RU}Z^{SV}\left(\eta^{TW} + \frac{2}{3}Z^{TW}\right) - \frac{i}{36}Z^{RSTUVW}\right)\Phi^*_{(R)}f_{RST}\Phi_{(U)}f_{UVW}\,, \tag{1.9}$$

where $Z^{RS} = Z_a{}^R Z_a{}^S$ and $Z^{RSTUVW} = \epsilon^{abcdef}Z_a{}^R Z_b{}^S Z_c{}^T Z_d{}^U Z_e{}^V Z_f{}^W$. The structure constants $f_{RST} = f_{RS}{}^V\eta_{VT}$ are totally antisymmetric.

2. The scalar potential: the SU(1,1)/U(1) scalars

In this section we will discuss properties of the scalar potential that are independent of the specific matter content and choice of gauge group. The potential can be written in the form:

$$V = \sum_{i,j}\left(R^{(ij)}\,V_{ij} + I^{(ij)}\,W_{ij}\right)\,. \tag{2.1}$$

The indices $i, j, \ldots$ label the different factors in the gauge group G, which we will take to be semi-simple. R and I contain the SU(1,1) scalars and depend on the gauge coupling constants and the SU(1,1) angles, V and W contain the structure constants, depend on the matter fields, and are symmetric resp. anti-symmetric in the indices i, j. We have:

$$\begin{aligned} R^{(ij)} &= \frac{g_i g_j}{2}(\Phi_i^* \Phi_j + \Phi_j^* \Phi_i) \\ &= g_i g_j \left(\cos(\alpha_i - \alpha_j) \frac{1+r^2}{1-r^2} - \frac{2r}{1-r^2} \cos(\alpha_i + \alpha_j + \varphi) \right) , \end{aligned} \tag{2.2}$$

$$\begin{aligned} I^{(ij)} &= \frac{g_i g_j}{2i}(\Phi_i^* \Phi_j - \Phi_j^* \Phi_i) \\ &= -g_i g_j \sin(\alpha_i - \alpha_j) . \end{aligned} \tag{2.3}$$

The fields r and φ represent the scalars of the SU(1,1)/ U(1) coset, we have solved the constraint (1.3) in a suitable U(1) gauge as

$$\phi_1 = \frac{1}{\sqrt{1-r^2}} , \qquad \phi_2 = \frac{r e^{i\varphi}}{\sqrt{1-r^2}} . \tag{2.4}$$

In this section we will discuss the extremum of this potential for r and φ. We introduce the quantities

$$C_\pm = \sum_{ij} g_i g_j \cos(\alpha_i \pm \alpha_j) V_{ij} , \qquad S_+ = \sum_{ij} g_i g_j \sin(\alpha_i + \alpha_j) V_{ij} , \tag{2.5}$$

$$T_- = \sum_{ij} g_i g_j \sin(\alpha_i - \alpha_j) W_{ij} , \tag{2.6}$$

and write the potential as

$$V = C_- \frac{1+r^2}{1-r^2} - \frac{2r}{1-r^2} \left(C_+ \cos\varphi - S_+ \sin\varphi \right) - T_- . \tag{2.7}$$

One finds that the extremum in φ is obtained for

$$\cos\varphi_0 = \frac{s_1 C_+}{\sqrt{C_+^2 + S_+^2}} , \qquad \sin\varphi_0 = -\frac{s_1 S_+}{\sqrt{C_+^2 + S_+^2}} , \quad (s_1 = \pm 1) . \tag{2.8}$$

The equation for the extremum in r becomes, for $\varphi = \varphi_0$,

$$r^2 + 1 - s_1 \frac{2 r C_-}{\sqrt{C_+^2 + S_+^2}} = 0 , \tag{2.9}$$

which, for

$$\Delta \equiv C_-^2 - C_+^2 - S_+^2 > 0 , \tag{2.10}$$

is solved by ($s_2 = \pm 1$, a priori independent of s_1):

$$r_0 = \frac{1}{\sqrt{C_+^2 + S_+^2}} \left(s_1 C_- + s_2 \sqrt{C_-^2 - C_+^2 - S_+^2} \right) . \tag{2.11}$$

Now consider the signs s_1 and s_2. The condition $r_0 < 1$ leads to

$$s_1 s_2 C_- + \sqrt{C_-^2 - C_+^2 - S_+^2} < 0\,, \tag{2.12}$$

which implies $s_1 s_2 = -\mathrm{sgn} C_-$. To have $r_0 \geq 0$ we need $s_1 = \mathrm{sgn} C_-$, so that $s_2 = -1$. After substitution of r_0 and φ_0 in V, we obtain

$$V_0 = \mathrm{sgn} C_- \sqrt{C_-^2 - C_+^2 - S_+^2} - T_-\,. \tag{2.13}$$

In the case that all SU(1,1) angles α_i vanish, $S_+ = T_- = 0$ and $C_- = C_+$, and (2.9) leads to $r_0 = 1$, which is a singular point of the parametrisation: there is no extremum. This is the generalisation of the Freedman-Schwarz potential [4] to the case of general matter coupling. The sign of the potential is the sign of C_-. We will not discuss this situation any further.

If $\Delta > 0$, which requires some of the SU(1,1) angles to be different, the extremum (2.13) exists and can be further simplified by looking in more detail at $C_\pm$, S_+. We find

$$\begin{aligned} C_-^2 - C_+^2 - S_+^2 &= \sum_{ij}\sum_{kl} g_i g_j g_k g_l\, V_{ij} V_{kl} (\cos(\alpha_i - \alpha_j)\cos(\alpha_k - \alpha_l) \\ &\quad - \cos(\alpha_i + \alpha_j)\cos(\alpha_k + \alpha_l) - \sin(\alpha_i + \alpha_j)\sin(\alpha_k + \alpha_l)) \\ &= 2\sum_{ij}\sum_{kl} g_i g_j g_k g_l\, V_{ij} V_{kl} \sin(\alpha_i - \alpha_k)\sin(\alpha_j - \alpha_l)\,. \end{aligned} \tag{2.14}$$

We see that the potential at the extremum in r, φ depends only on the combinations $g_i g_j \sin(\alpha_i - \alpha_j)$. This was known for gauged supergravity without additional matter, and now turns out to be a general property.

If the condition (2.10) holds we find an extremum for the SU(1,1) scalars. To see what happens at the extremum it is useful to work out the second derivatives of the potential. We will do this with respect to the variables

$$x = r\cos\varphi\,, \qquad y = r\sin\varphi\,. \tag{2.15}$$

One easily finds that in the extremum

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_0 = \left.\frac{\partial^2 V}{\partial y^2}\right|_0 = \frac{4}{(1-r_0^2)^2}\,\mathrm{sgn} C_- \sqrt{C_-^2 - C_+^2 - S_+^2}\,, \qquad \left.\frac{\partial^2 V}{\partial x \partial y}\right|_0 = 0\,. \tag{2.16}$$

If $\mathrm{sgn} C_- > 0\,(< 0)$ the potential has a minimum (maximum) for the scalars x, y. We see that, up to a scale factor, the second derivatives are equal to the first term in the potential at the extremum (2.13). The scale factor can be understood by considering the kinetic term for the scalars. It can be read off from (1.2), and, after expressing the action in terms of the variables x, y we find:

$$\mathcal{L}_{\mathrm{kin},\phi} = -\frac{2}{(1-r^2)^2}\Big(\partial_\mu x \partial^\mu x + \partial_\mu y \partial^\mu y\Big)\,. \tag{2.17}$$

To have a proper normalisation for these fields requires a rescaling:

$$x = \frac{1}{2} x' (1 - r_0)^2\,, \qquad y = \frac{1}{2} y' (1 - r_0)^2\,. \tag{2.18}$$

The contribution to the action of the new scalars x', y', including the potential, then looks as follows:

$$\mathcal{L}_\phi = -\frac{1}{2}\left(\frac{1-r_0^2}{1-r^2}\right)^2 \left(\partial_\mu x' \partial^\mu x' + \partial_\mu y' \partial^\mu y'\right) - \mathrm{sgn} C_- \sqrt{C_-^2 - C_+^2 - S_+^2} + T_- \\ -\frac{1}{2}\mathrm{sgn} C_- \sqrt{C_-^2 - C_+^2 - S_+^2}\left(x'^2 + y'^2\right) + \ldots . \tag{2.19}$$

We find that in these variables the equality $V_0 = m^2$ is exact, except for the term T_- (2.6). The reason is of course that T_- only depends on the matter scalars and therefore does not contribute to the second derivatives. The presence of T_- therefore violates the equality (1.1).

In (2.19) we also see that the (mass)^2 is determined by the sign of C_-. The sign of C_- also determines the sign of the first term in the extremum of the potential. If we go back to the potential itself (2.7), we see that for $r \to 1$ the potential behaves as

$$V \to \frac{1}{1-r}\big(C_- - C_+ \cos\varphi + S_+ \sin\varphi\big) . \tag{2.20}$$

Therefore along the unit circle the potential goes to infinity, and the sign in this limit, assuming $\Delta > 0$, is again the same as $\mathrm{sgn} C_-$. In the origin $r \to 0$ the potential and its derivatives are well behaved.

We conclude that if (2.10) holds and if $\mathrm{sgn} C_- > 0$, the potential as a function of the SU(1,1) scalars has a minimum, and goes to $+\infty$ along the unit circle. The value of the potential gets a contribution which has the same sign as C_-, but the value and sign of the minimum depends also on T_-. In the next sections we will try to find examples with this behaviour.

3. The scalar potential: matter multiplets

The $N = 4$ vector multiplet contains one vector and six scalar fields, and, in the construction of $N = 4$ Poincaré supergravity with the superconformal method, six multiplets are required to gaugefix the superconformal symmetries. Adding $n + 6$ vector multiplets to conformal supergravity thus gives n matter multiplets, and $6n$ physical scalars. These are however expressed in terms of $6n + 36$ variables (the $Z_a{}^R$), of which 36 are eliminated by gaugefixing the local SO(6) symmetry and by the constraint (1.4). The constraint implies that Z corresponds to the first six rows of an SO$(6, n)$ element.

The constraint is a complication in the analysis of the scalar sector of the theory. Fortunately, there is a nice way to parametrise the solution of the constraint, as was remarked in [14].[3] The dimensional reduction of the $D = 10$, $N = 1$ supergravity theory, coupled to m vector multiplets, gives after reduction $12 + m$ vectors in $D = 4$. The scalars resulting from this reduction are the 21 scalars coming from the $D = 10$ metric, the 15 scalars coming from the $D = 10$ two-form, and $6m$ scalars from the $D = 10$ vector

[3] Other parametrisations are discussed in [8]

multiplets, altogether $6m+36$. These scalars parametrise the coset $\mathrm{SO}(6,6+m)/\mathrm{SO}(6)\times\mathrm{SO}(6+m)$ [15]:

$$N=\begin{pmatrix} G^{-1} & G^{-1}(B+W) & \sqrt{2}G^{-1}U \\ (-B+W)G^{-1} & (G-B+W)G^{-1}(G+B+W) & \sqrt{2}(G-B+W)G^{-1}U \\ \sqrt{2}U^TG^{-1} & \sqrt{2}U^TG^{-1}(G+B+W) & \mathbb{1}_m+2U^TG^{-1}U \end{pmatrix}, \tag{3.1}$$

where $G(B)$ are (anti)-symmetric 6×6 matrices, U is a $6\times m$ matrix, and $W=U^TU$. The matrix N satisfies

$$N\gamma N^T=\gamma\,, \tag{3.2}$$

with

$$\gamma=\begin{pmatrix} 0 & \mathbb{1}_6 & 0 \\ \mathbb{1}_6 & 0 & 0 \\ 0 & 0 & \mathbb{1}_m \end{pmatrix}. \tag{3.3}$$

We now transform the metric γ to the metric η, and N to N' by

$$\eta=M\gamma M^T\,,\qquad N'=MNM^T\,, \tag{3.4}$$

with

$$M=\frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_6 & -\mathbb{1}_6 & 0 \\ -\mathbb{1}_6 & -\mathbb{1}_6 & 0 \\ 0 & 0 & \mathbb{1}_m \end{pmatrix}. \tag{3.5}$$

From the matrix N' we can read off what Z_a^R is: the first six rows N'. Thus we have an explicit parametrisation of the ${Z_a}^R$ for $n=6+m$.

We emphasize however that this does not imply that the gauged $D=4$ theory with arbitrary $\mathrm{SU}(1,1)$ angles follows by reduction from $D=10$. We only use the reduction from $D=10$ to solve the constraints.

In this paper we will limit ourselves to the case where $m=0$, or $U=0$ in (3.1). This corresponds to six vector multiplets added to Poincaré supergravity. We split the indices $R,S,\ldots$ of η_{RS} in $A,B,\ldots=1,\ldots,6$, ($\eta_{AB}=-\delta_{AB}$) and $I,J,\ldots=7,\ldots,12$, ($\eta_{IJ}=+\delta_{IJ}$). The scalar constraint (1.4) then reads

$$XX^T-YY^T=\mathbb{1}_6\,, \tag{3.6}$$

where ${X_a}^A={Z_a}^A$, ${Y_a}^I={Z_a}^I$. X and Y are both 6×6 matrices, which together form the first six rows of the matrix N':

$$X=\frac{1}{2}\left(G+G^{-1}+BG^{-1}-G^{-1}B-BG^{-1}B\right), \tag{3.7}$$
$$Y=\frac{1}{2}\left(G-G^{-1}-BG^{-1}-G^{-1}B-BG^{-1}B\right). \tag{3.8}$$

The scalar potential will depend on $Z^{AB}=(X^TX)^{AB}$, $Z^{AI}=(X^TY)^{AI}$ and $Z^{IJ}=(Y^TY)^{IJ}$.

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In the examples in section 4 we will further simplify matters by choosing for G and B:

$$G = \begin{pmatrix} a\mathbb{1}_3 & 0 \\ 0 & a\mathbb{1}_3 \end{pmatrix} \quad (a > 0)\,, \qquad B = \begin{pmatrix} 0 & b\mathbb{1}_3 \\ -b\mathbb{1}_3 & 0 \end{pmatrix}\,, \tag{3.9}$$

which gives

$$X = \frac{a^2+b^2+1}{2a}\mathbb{1}_6\,, \qquad Y = \frac{1}{2a}\begin{pmatrix} (a^2+b^2-1)\mathbb{1}_3 & -2b\mathbb{1}_3 \\ 2b\mathbb{1}_3 & (a^2+b^2-1)\mathbb{1}_3 \end{pmatrix}\,. \tag{3.10}$$

The variables Z^{RS} are then easily determined to be:

$$\begin{aligned} Z^{AB} &= \frac{(a^2+b^2+1)^2}{4a^2}\mathbb{1}_6\,, \\ Z^{AI} &= \frac{a^2+b^2+1}{4a^2}\begin{pmatrix} (a^2+b^2-1)\mathbb{1}_3 & -2b\mathbb{1}_3 \\ 2b\mathbb{1}_3 & (a^2+b^2-1)\mathbb{1}_3 \end{pmatrix}\,, \\ Z^{IJ} &= \frac{1}{4a^2}\Big((a^2+b^2+1)^2 - 4a^2\Big)\mathbb{1}_6\,. \end{aligned} \tag{3.11}$$

4. Examples

With the ingredients of sections 2 and 3 we will now work out a number of examples. We have seen that the analysis of the SU(1,1) scalars in section 2 depends crucially on condition (2.10), $\Delta > 0$. We will only work out cases for which this condition is satisfied for all values of the matter fields a and b in our parametrisation of G and B (3.9). This excludes among others potentials of the Freedman-Schwarz type [4]. To evaluate (2.10) we will need the contributions V_{ij} and W_{ij} to the potential, see (2.1). These are given by

$$V_{ij} = \frac{1}{4}Z^{RU}Z^{SV}\,(\eta^{TW} + \frac{2}{3}Z^{TW})\, f^{(i)}{}_{RST}f^{(j)}{}_{UVW}\,, \tag{4.1}$$

$$W_{ij} = \frac{1}{36}\epsilon^{abcdef}Z_a{}^R Z_b{}^S Z_c{}^T Z_d{}^U Z_e{}^V Z_f{}^W\, f^{(i)}{}_{RST}f^{(j)}{}_{UVW}\,, \tag{4.2}$$

where the $f^{(i)}$ are the structure constants for the different factors of the semi-simple gauge group.

4.1 SO(3)^4

First we will consider a product of compact groups: $G = \mathrm{SO}(3)^4$. For the group SO(3) the structure constants are

$$f_{RS}{}^T = -\epsilon_{RST}\,, \qquad f_{RST} = f_{RS}{}^U\eta_{UT}\,. \tag{4.3}$$

The sign of f_{RST}, which is completely anti-symmetric, therefore depends on the sign of the element η_{TT}. The four SO(3) groups are labelled $i = 1, \ldots, 4$, and associated with the values $R, S, \ldots$ in the following way:

$$R, S, \ldots = \overbrace{1\;2\;3}^{i=1}\;\overbrace{4\;5\;6}^{i=2}\;\overbrace{7\;8\;9}^{i=3}\;\overbrace{10\;11\;12}^{i=4}\,. \tag{4.4}$$

We then obtain the contributions V_{ij} and W_{ij} to the potential:

$$
\begin{aligned}
V_{11} &= V_{22} = \frac{1}{64a^6}\left(1+a^2+b^2\right)^4\left(\left(1+a^2+b^2\right)^2-6a^2\right),\\
V_{33} &= V_{44} = \frac{1}{64a^6}\left(\left(1+a^2+b^2\right)^2-4a^2\right)^2\left((1+a^2+b^2)^2+2a^2)\right),\\
V_{12} &= V_{34} = 0\,,\\
V_{13} &= V_{24} = -\frac{1}{64a^6}\left(a^2+b^2+1\right)^3\left(a^2+b^2-1\right)^3,\\
V_{23} &= -V_{14} = -\frac{1}{64a^6}\left(a^2+b^2+1\right)^3(2b)^3\,,\\
W_{12} &= \frac{1}{64a^6}\left(1+a^2+b^2\right)^6,\\
W_{34} &= \frac{1}{64a^6}\left(\left(1+a^2+b^2\right)^2-4a^2\right)^3,\\
W_{14} &= -W_{23} = -\frac{1}{64a^6}\left(a^2+b^2+1\right)^3\left(a^2+b^2-1\right)^3,\\
W_{13} &= W_{24} = -\frac{1}{64a^2}\left(a^2+b^2+1\right)^3(2b)^3\,.
\end{aligned}
\tag{4.5}
$$

Using these, we will discuss the properties of the potential for some special cases:

- $g_3 = g_4 = 0$. We assume $\alpha_1 \neq \alpha_2$. Then $C_- = g_1^2 V_{11} + g_2^2 V_{22} = V_{11}(g_1^2 + g_2^2)$. V_{11} is negative inside the circle

$$\left(a-\frac{1}{2}\sqrt{6}\right)^2 + b^2 = \frac{1}{2}\,. \tag{4.6}$$

  We find

$$\Delta = 4V_{11}^2 g_1^2 g_2^2 \sin^2(\alpha_1-\alpha_2) \tag{4.7}$$

  which is never negative: the extremum for the $\mathrm{SU}(1,1)$ scalars discussed in section 2 exists for all values of a and b if $\Delta > 0$, on the circle (4.6) $V_{11} = C_\pm = S_+ = \Delta = 0$ and the potential (2.7) is independent of r and φ. The matter potential in the extremum of the $\mathrm{SU}(1,1)$ scalars reads

$$V = V_{11}|2g_1g_2\sin(\alpha_1-\alpha_2)| - 2g_1g_2\sin(\alpha_1-\alpha_2)W_{12}\,. \tag{4.8}$$

  Consider the case $g_1g_2\sin(\alpha_1-\alpha_2) < 0$. The matter potential then reads:

$$
\begin{aligned}
V &= 2|g_1g_2\sin(\alpha_1-\alpha_2)|\left(V_{11}+W_{12}\right)\\
&= 2|g_1g_2\sin(\alpha_1-\alpha_2)| \times \frac{1}{32a^6}\left(1+a^2+b^2\right)^4\left(\left(a^2+b^2+1\right)^2-3a^2\right),
\end{aligned}
\tag{4.9}
$$

  which is positive for all values of a and b. The extremum is reached for $a = 1,\ b = 0$ corresponding to $V_0 = |g_1g_2\sin(\alpha_1-\alpha_2)|$. This is obviously a minimum in the a, b variables, but, since at this point $C_- < 0$, a maximum in the $\mathrm{SU}(1,1)$ scalars. Note that the positivity of the extremum is due to T_-. For values of a and b outside the circle (4.6) the $\mathrm{SU}(1,1)$ scalars have a minimum. In the case $g_1g_2\sin(\alpha_1-\alpha_2) > 0$ the potential is negative everywhere.

- $g_1 = g_2 = 0$. In this case $C_- = (g_3^2 + g_4^2)V_{33}$ and Δ are proportional to V_{33}, which vanishes for $a = 1,\ b = 0$, and is positive elsewhere. The analysis of section 2 is therefore valid for all a and b, and gives a minimum in the $r,\ \varphi$ variables, except in the point $a = 1,\ b = 0$, where the potential (2.7) and Δ vanish. This case corresponds to two SO(3) Yang-Mills multiplets coupled to ungauged supergravity. The absolute minimum at $a = 1,\ b = 0$ corresponds to vanishing cosmological constant. It is not difficult to see that this is the only extremum of the potential for this particular choice of the matter sector.

- $g_2 = g_4 = 0$ etc. In all cases where we take one SO(3) from the supergravity ($R, S = 1, \ldots, 6$) and one SO(3) from the matter sector there are regions in a and b with $\Delta < 0$.

4.2 $\mathrm{SO}(3)^2 \times \mathrm{SO}(2,1)^2$

For the noncompact group SO(2,1) we need to assign three values of the indices $R, S, \ldots$ corresponding to different values of the diagonal metric η. We choose the groups in the following way:

$$R, S, \ldots = \overbrace{1\ 2\ 7}^{i=1}\ \overbrace{4\ 5\ 6}^{i=2}\ \overbrace{3\ 8\ 9}^{i=3}\ \overbrace{10\ 11\ 12}^{i=4}\,. \tag{4.10}$$

The groups labelled 1 and 3 therefore correspond to SO(2,1). The SO(2,1) structure constants are chosen as

$$f_{RS}{}^{T} = -\epsilon_{RSU}\eta^{UT}\,, \qquad f_{RST} = -\epsilon_{RST}\,. \tag{4.11}$$

It should be noted that the form of V_{ij} and W_{ij}, and therefore the resulting examples, depends crucially on the way the groups are distributed over the 12 vector multiplets in (4.10). The analysis in this and the following subsection is therefore far from exhaustive.

The contributions V_{ij} and W_{ij} to the potential now take on the form:

$$\begin{aligned}
V_{11} &= \frac{1}{16a^6}\left(a^2+b^2+1\right)^2\left(b^2\left(a^2+b^2+1\right)^2 + 2a^2\left(a^2-b^2\right)\right),\\
V_{22} &= \frac{1}{64a^6}\left(a^2+b^2+1\right)^4\left(\left(a^2+b^2+1\right)^2 - 6a^2\right),\\
V_{33} &= \frac{1}{16a^6}\left(b^2\left(a^2+b^2+1\right)^4 + 2a^2\left(a^2-b^2\right)\left(a^2+b^2+1\right)^2 - 8a^6\right)\\
V_{44} &= \frac{1}{64a^6}\left(\left(a^2+b^2+1\right)^2 - 4a^2\right)^2\left((a^2+b^2+1)^2 + 2a^2\right),\\
V_{12} &= V_{13} = V_{23} = V_{34} = V_{14} = 0\,,\\
V_{24} &= -\frac{1}{64a^6}\left(a^2+b^2+1\right)^3\left(a^2+b^2-1\right)^3,\\
W_{12} &= W_{14} = W_{23} = W_{34} = 0\,,\\
W_{13} &= W_{24} = -\frac{1}{8a^6}b^3\left(a^2+b^2+1\right)^3.
\end{aligned} \tag{4.12}$$

In the search for interesting examples with just two nonzero coupling constants g_i and g_j we find that

$$\Delta = \left(2g_ig_j\sin(\alpha_i-\alpha_j)\right)^2\left(V_{ii}V_{jj} - V_{ij}^2\right). \tag{4.13}$$

Except for the case $i = 2,\ j = 4$, which concerns the two compact subgroups and was already treated in section 4.1, the off-diagonal entries in V vanish, and the requirement $\Delta > 0$ implies that for both subgroups the corresponding V_{ii} must be nonnegative. With the present choice of subgroups this means that $i = 2$ can be discarded. We find that both V_{33} and V_{44} are positive, except in the point $a = 1,\ b = 0$, where they vanish. V_{11} is positive everywhere, with a minimum at $a = 1,\ b = 0$ with value $\frac{1}{2}$. Since also W_{ij}=0, either everywhere or at the point $a = 1,\ b = 0$, the second term in the potential, T_-, vanishes and will not help in obtaining a positive cosmological constant.

The result with two groups will therefore always be such that the extremum of the potential is at $a = 1,\ b = 0$, and that the potential vanishes there.

4.3 $\mathrm{SO}(2,1)^4$

The structure constants are as in the section 4.2. The groups are now assigned in the following way:

$$R, S, \ldots = \overbrace{1\ 2\ 7}^{i=1}\ \overbrace{4\ 5\ 10}^{i=2}\ \overbrace{3\ 8\ 9}^{i=3}\ \overbrace{6\ 11\ 12}^{i=4}\,. \tag{4.14}$$

The results for the potential do depend on how the groups are assigned, clearly different choices are possible. The results with the choice above are:

$$\begin{aligned}
V_{11} &= V_{22} = \frac{1}{16a^6}\left(a^2+b^2+1\right)^2\left(b^2\left(a^2+b^2+1\right)^2+2a^2\left(a^2-b^2\right)\right),\\
V_{33} &= V_{44} = \frac{1}{16a^6}\left(b^2\left(a^2+b^2+1\right)^4+2a^2\left(a^2-b^2\right)\left(a^2+b^2+1\right)^2-8a^6\right),\\
V_{ij} &= 0 \text{ for } i \neq j\,,\\
W_{12} &= W_{34} = 0\,,\\
W_{13} &= W_{24} = -\frac{1}{8a^6}b^3\left(a^2+b^2+1\right)^3,\\
W_{14} &= -W_{23} = -\frac{1}{16a^6}b^2\left(a^2+b^2-1\right)\left(a^2+b^2+1\right)^3.
\end{aligned} \tag{4.15}$$

The interesting case here is

- $g_3 = g_4 = 0$. Now we have $C_- = V_{11}(g_1^2 + g_2^2)$, which is always positive, and

$$V = V_{11}\, 2|g_1 g_2 \sin(\alpha_1 - \alpha_2)|\,. \tag{4.16}$$

  The potential is everywhere positive, the condition $\Delta > 0$ is trivially satisfied, and $C_- > 0$. There is therefore an absolute minimum in all variables r, φ, a and b. In this case there is no contribution from T_-, and therefore (1.1) will be satisfied. We will come back to further properties of this example in section 5

5. Implications and discussion

We have analysed some special cases of gauged $N = 4$ supergravity with six additional matter multiplets with the aim of finding solutions with positive cosmological constant. Two situations arise in which this might happen: one is the example of $\mathrm{SO}(3)^4$, where a saddle point is found for a positive value of the potential: the extremum is a minimum in the two matter fields considered, but a maximum in the two $\mathrm{SU}(1,1)$ scalars from the supergravity sector. This is an extension with matter multiplets of the potentials studied by [5].

Another case involves $\mathrm{SO}(2,1)^2$, where we find a positive minimum of the potential in all variables. However, in this case the gauge group is noncompact, which might give rise to wrong-sign kinetic terms. The kinetic term of the $\mathrm{SU}(1,1)$ scalars was given in (2.19) and does not have this problem. Also the kinetic term of the matter scalars is independent of the gauging. For the fields a and b one finds

$$\begin{aligned}\mathcal{L}_{\mathrm{kin}\,a,b} = -\frac{3}{4a^4}\Big(& 4a^2\left(1+b^2\right)(\partial a)^2 + \left(\left(a^2+b^2+1\right)^2-4a^2b^2\right)(\partial b)^2 \\ & + 4ab\left(-a^2+b^2+1\right)\partial a\partial b\Big)\,.\end{aligned} \tag{5.1}$$

After diagonalizing this to new fields a' and b' one obtains

$$\mathcal{L}_{\mathrm{kin}\,a,b} = -\frac{3}{4a^4}\left(4a^2(\partial a')^2 + \left(1+a^2+b^2\right)^2(\partial b')^2\right), \tag{5.2}$$

so these kinetic terms have indeed the standard sign for all a,b. In the extremum $a = 1,\ b = 0$ one finds

$$\mathcal{L}_{\mathrm{kin}\,a,b} \to -3\left((\partial a)^2 + (\partial b)^2\right). \tag{5.3}$$

The potential and its second derivatives for this example are in the extremum

$$V(a,b) = |g_1 g_2 \sin(\alpha_1-\alpha_2)|\left(1+(a-1)^2+2b^2+\ldots\right)\,. \tag{5.4}$$

For these scalars the relation between the potential and its second derivatives is not satisfied, since after a rescaling of a and b to get the standard normalisation of the kinetic terms the $(\mathrm{mass})^2$ of a and b still differ by a factor 2, and are not equal to V_0.

The vector kinetic terms do depend on the gauging, and this might cause a problem for noncompact groups. The vector kinetic terms are for $a=1,\ b=0$,

$$-\frac{1}{4}\eta_{RS}F^{+\,R}_{\mu\nu}F^{\mu\nu\,+\,S}\left(\frac{1}{\Phi_{(R)}}\left(\phi^1_{(R)}-\phi^2_{(R)}\right)-\frac{2}{|\Phi_{(R)}|^2}\right)+\text{h.c.}\,. \tag{5.5}$$

In the case the gauging is $\mathrm{SO}(3)^2$, as in the first example of section 4.1, this is the form of the kinetic terms at the extremum of the matter fields. The sum over R,S then runs over the indices $1,\ldots,6$, and the kinetic terms have the standard signs, as was shown in detail in [7]. In the example of section 4.3 the form of the kinetic terms in the matter extremum is again (5.5), but now the indices run over the values $1,2,7,4,5,10$ and two of the kinetic terms change sign. This is a problem for the solution of section 4.3.

In [13] it was found that potential and its second derivatives in extended supergravity theories satisfies (1.1). In section 2 we clarified this relation for the $\mathrm{SU}(1,1)$ scalars. We have shown that (1.1) is modified by the term T_- in the scalar potential, but that for $T_-=0$ the relation is indeed valid. As we have seen in this section, the matter scalars may violate (1.1). The relation (1.1) is reminiscent of supersymmetry Ward identities used in the past to investigate cases with partially broken supersymmetry [16]. It would be interesting to make (1.1) more precise in this way.

Much work remains to be done to give a more complete analysis of the matter sector with six or more multiplets, and of the examples that we have presented in this paper. Also their possible application in cosmological models remains to be studied. This is not only true for the example with an absolute minimum, but equally so for cases with tachyonic modes.

Acknowledgments

It is a pleasure to thank Ulf Gran and Peter Wagemans for useful discussions. The work of DBW is part of the research programme of the "Stichting voor Fundamenteel Onderzoek van de Materie" (FOM). This work is supported in part by the European Commission RTN programme HPRN-CT-2000-00131, in which MdR and DBW are associated to the University of Utrecht. SP thanks the University of Groningen for hospitality.

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